Generalized Heisenberg relation and Quantum Harmonic Oscillators

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Abstract

We study the consequences of the generalized Heisenberg uncertainty relation which admits a minimal uncertainty in length such as the case in a theory of quantum gravity. In particular, the theory of quantum harmonic oscillators arising from such a generalized uncertainty relation is examined. We demonstrate that all the standard properties of the quantum harmonic oscillators prevail when we employ a generalized momentum. We also show that quantum electrodynamics and coherent photon states can be described in the familiar standard manner despite the generalized uncertainty principle.

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I. INTRODUCTION

We begin with the generalized commutation relation,

$$[q_{\mathbf{k},i}, p_{\mathbf{k}',j}] = i\hbar\delta_{\mathbf{k},\mathbf{k}'}\,\delta_{i,j}\left(1 + \beta p_{\mathbf{k},i}\,p_{\mathbf{k}',j}\right),\tag{1}$$

which reduces to the fundamentally simple form if we consider a single degree of freedom,

$$[q,p] = i\hbar(1 + \beta p^2).\tag{2}$$

This leads to the generalized Heisenberg uncertainty relation

$$\Delta q\Delta p \geq \frac{1}{2}\hbar\left(1 + \beta(\Delta p)^2\right),\tag{3}$$

where β is positive and independent of $\Delta q, \Delta p$. The above relation implies [1] a minimum length described by the parameter β. This type of generalized uncertainty relation has appeared in the context of quantum gravity and string theory and is based on the premise that a minimal length should quantum theoretically be described as a minimal uncertainty in position measurements. Several consequences of the above relations have been of much interest [1–5] recently. This has lead to a great deal of interest in several aspects such as the Hilbert space structure of the theory in the presence of quantum gravity, the formulation of Liouville theorem in statistical physics, the cosmological constant problem, the deformation of the black body radiation, consequences in quantum electrodynamics in the presence of quantum gravity and deformed local current algebra compatible with a fundamental length scale. We shall, for the sake of expediency, regard the generalized commutation relation as due to a deformation, as a result of quantum gravity and accordingly refer to the system obeying Eq.(3) as deformed; we shall refer to the limit $\beta \to 0$ simply as the undeformed case. Our goal here is a modest investigation of the question: what are the consequences when the physical system is viewed as a set of quantum harmonic oscillators?

We shall consider single level oscillators, for convenience. The paper is organized as follows. Section I begins with a summary of known results and formulates the algebra and the Fock space of the quantum oscillators in the presence of deformation. The ground state is determined in Section II. We study the Heisenberg equations of motion in Section III and demonstrate in particular that Liouville theorem prevails in the standard form despite the presence of deformation when the system is described by generalized momenta. Section IV is devoted to summarizing the results in quantum electrodynamics and quantum optics to show that consequences of deformation leading to a generalized Heisenberg uncertainty principle are not explicitly present. Section V concludes with a summary.

Let us proceed by introducing the harmonic oscillators by means of the operators,

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(q + \frac{iP}{m\omega}\right)\quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(q - \frac{iP}{m\omega}\right),\tag{4}$$

where q is the coordinate and P is the generalized momentum whose properties will be described below. We can choose

$$P = p + f(p),\tag{5}$$

where $f(p)$ is a general function which must be so chosen in order that it will lead to simple and desirable properties for the system of harmonic oscillators. We may in fact, choose,

$$f(p) = \sum_{r=1}^{\infty} \frac{(-\beta)^r}{2r+1} \, p^{2r+1} \tag{6}$$

for this reason. We shall adhere to the suggestion of Camacho [3] who has shown that this choice leads to the uncertainty relation in Eq.(3), as well as to the further useful result

$$[q, f(p)] = -i\hbar\beta p^2 \,, \tag{7}$$

and consequently we obtain

$$[q, P] = i\hbar \,. \tag{8}$$

In other words, the choice

$$P = p + f(p) = \frac{1}{\sqrt{\beta}} \arctan(\sqrt{\beta}p) \,, \tag{9}$$

determines P so that it plays the role of a generalized momentum in the sense of the canonical commutation relation, Eq.(8). Consequently, from the representation in Eq.(4), we obtain

$$aa^{\dagger} = \frac{1}{\hbar\omega} H + \frac{1}{2}, \quad a^{\dagger}a = \frac{1}{\hbar\omega} H - \frac{1}{2}, \tag{10}$$

where

$$H = \frac{1}{2}m\omega^2 q^2 + \frac{P^2}{2m} \tag{11}$$

is the Hamiltonian in terms of the generalized momentum P. We immediately obtain the result

$$aa^{\dagger} - a^{\dagger}a = 1 \,. \tag{12}$$

This is the standard algebra of oscillators, a remarkable result [3] despite the deformation introduced as in Eq.(3). In other words, the standard algebra of harmonic oscillators prevails despite the deformation due to quantum gravity. To develop this idea further, we proceed as follows. We build the Fock states from the ground state by the construction

$$|n\rangle = \frac{(a^{\dagger})^n}{\sqrt{n!}} \, |0\rangle \,. \tag{13}$$

Let us define the operator relation $a^{\dagger}a = N$, the number operator which in turn leads to $aa^{\dagger} = N+1$ as a consequence of the algebra above. The number operator acts on the states so that $N|n\rangle = n|n\rangle$, defining the Fock states $|n\rangle$, where $n = 0, 1, 2, \cdots$. The operators $a, a^{\dagger}$ are seen to satisfy the relations

$$[N, a] = -a, \quad [N, a^{\dagger}] = a^{\dagger} \,. \tag{14}$$

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By examining $Na^\dagger|n\rangle$, we conclude in the standard manner that $a^\dagger$ is the creation operator and a is the annihilation operator. Indeed we find,

$$a|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{15}$$

which is no different from the standard results of the undeformed oscillator. All this agrees with the well-known properties of the operators for the standard system of quantum harmonic oscillators. What is significant now is that these results are valid for the case when the Heisenberg commutation relation is generalized to identify a fundamental length in the uncertainty principle arising from a deformation such as in quantum gravity. From Eq.(11), we further obtain the important relation

$$H = \frac{1}{2}\hbar\omega(a^\dagger a + aa^\dagger) = \hbar\omega(N + \frac{1}{2})\,, \tag{16}$$

and hence the energy spectrum is given by

$$E_n = \hbar\omega(n + \frac{1}{2})\,, \tag{17}$$

when $n = 0, 1, \cdots$, for the quantum system with the generalized Heisenberg relation when the Hamiltonian is given by Eq.(12). Despite the introduction of a fundamental length, the standard properties of the harmonic oscillators thus prevail, and so does the energy spectrum. Here the generalized momentum may be expressed in the form of a series,

$$P = \frac{1}{\sqrt{\beta}}\arctan(\sqrt{\beta}p) = \frac{1}{\sqrt{\beta}}\left\{\sqrt{\beta}p - \frac{(\sqrt{\beta}p)^3}{3} + \frac{(\sqrt{\beta}p)^5}{5} + \cdots\right\} \tag{18}$$

when $\beta \neq 0$, and we recover the ordinary momentum in the limiting case:

$$\lim_{\beta\longrightarrow 0} P = p\,. \tag{19}$$

The square of the momentum occurring in the Hamiltonian may be identified by the series

$$P^2 = p^2 - \frac{2\beta p^4}{3} + \frac{23\beta^2 p^6}{45} - \frac{44\beta^3 p^8}{105} + \frac{563\beta^4 p^{10}}{1575} + \cdots\,. \tag{20}$$

We expect that this generalized momentum is related to the coordinate in the form $P = -i\hbar(\partial/\partial_q)$. Indeed this relation is true up to a constant phase as pointed out in Dirac's textbook [6,7]. This introduces a tremendous simplification in the theory. We may consider the ground state, defined by

$$a\psi_0 = (q + \frac{iP}{m\omega})\,\psi_0 = 0\,, \tag{21}$$

in terms of the coordinate x and the generalized momentum P. This leads to the standard harmonic oscillator with the solution

$$\psi_0 = (\frac{m\omega}{\pi\hbar})^{1/4}\,e^{-m\omega^2/2\hbar}\,. \tag{22}$$

The other states can similarly be shown to correspond to the case of the standard harmonic oscillators. It must be pointed out that the Hamiltonian in Eq.(11) involves the momentum P which has lead to the great simplification. Thus although the form of the Hamiltonian is the same, the harmonic oscillator considered here differs from that of references [1] and [2].

II. EQUATIONS OF MOTION

We shall now investigate the Heisenberg equations of motion for operators in the form,

$$i\hbar\dot{F} = [F, H] \,, \tag{23}$$

where $\dot{F}$ is the full time derivative of the operator F and we shall assume no explicit time dependence. In particular , let us consider the relation

$$i\hbar\dot{a} = [a, H] = \sqrt{\frac{m\omega}{2\hbar}} \left[q + \frac{iP}{m\omega} \,, H \right] \,. \tag{24}$$

We can evaluate the commutator $[a, H]$ either directly or by making use of the form in Eq.(16) and the result is

$$\dot{a} = -i\omega a \,. \tag{25}$$

Hence we recover the familiar form of time dependence of the annihilation operator in a self-consistent manner:

$$a(t) = a_0 e^{-i\omega t} \,. \tag{26}$$

Let us next consider

$$i\hbar\dot{p} = [p, H] \,. \tag{27}$$

Evaluating in a straight forward manner, we obtain

$$\dot{p} = -m\omega^2(1 + \beta p^2)q \,, \tag{28}$$

which reduces to the standard result in the limit when β vanishes. Furthermore, we obtain

$$\dot{P} = -m\omega^2 q \,, \tag{29}$$

a more interesting result for the generalized momentum. Finally, from

$$i\hbar\dot{q} = [q, H] \,, \tag{30}$$

we obtain the result

$$\dot{q} = \frac{P}{m} = \frac{1}{m\sqrt{\beta}} \arctan(\sqrt{\beta}p) \,, \tag{31}$$

which possesses the property $\lim_{\beta \longrightarrow 0} \dot{q} = p/m$. This can be expressed as a series in powers of β. However, we may prefer the direct form in terms of the the generalized momentum P: this formulation seems more direct and simple when expressed in terms of the generalized momentum.

It can be further verified that the Hamilton's equations are valid in their canonical form:

$$\dot{q}_i = -\frac{i}{\hbar}\,[q_i, H] = \partial_{P_i} H \quad \dot{P}_i = -\partial_{q_i} H \,. \tag{32}$$

Accordingly, the momentum P is related to p by a canonical transformation. In this form, it is interesting to observe the quantum analog of the classical equations of motion, in particular [9] the correspondence:

$$\{A,\, B\} \implies -\frac{i}{\hbar}[A,\, B]\,, \tag{33}$$

where the braces represent the Poisson bracket

$$\{A, B\} = \sum_{i=1}^{3N}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial P_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial P_i},\right) \tag{34}$$

in terms of q_i the coordinate and P_i the generalized momentum.

We shall now derive the Liouville theorem of statistical mechanics and convince ourselves that it is valid in the standard form even in the deformed case when expressed in terms of the momentum P. Thus, beginning from the continuity relation valid in fluid mechanics,

$$\int_\omega d\omega \,\left(\frac{\partial}{\partial t}\rho(q_i, P_i) + \nabla\cdot\rho\,\mathbf{v}\right) = 0\,, \tag{35}$$

where $\rho(q_i, P_i)$ is the phase space density in the space of $3N$ dimensions, $i = 1,\,\cdots\,3N$ where the phase space is made of the coordinates and the generalized momenta. We then express the flux rate in the form

$$\nabla\cdot\rho\,\mathbf{v} = \sum_{i=1}^{3N}\left\{\frac{\partial\rho}{\partial q_i}\,\dot{q}_i + \frac{\partial\rho}{\partial P_i}\,\dot{P}_i + \rho\,\frac{\partial\dot{q}_i}{\partial q_i} + \rho\,\frac{\partial\dot{P}_i}{\partial P_i}\right\}\,, \tag{36}$$

and accordingly,

$$\nabla\cdot\rho\,\mathbf{v} = \{\rho,\, H\} \tag{37}$$

We may now employ the Hamilton equations, and accordingly arrive at the result,

$$\frac{d\rho(q_i, P_i)}{dt} = \frac{\partial\rho}{\partial t} + \{\rho,\, H\} = 0\,, \tag{38}$$

i.e., the total time derivative of the phase space density vanishes as the system evolves along a phase space trajectory, which is Liouville theorem in phase space. Consequently, the theorem prevails in its standard form even in the deformed case, such as when the deformation arises from quantum gravity.

III. PHOTONS

In the case of quantum electrodynamics in the presence of deformation, the Hamiltonian would be given by

$$H = \int d^3r \left(2\pi c^2 |P|^2 + \frac{1}{8\pi} (\nabla \times \mathbf{A})^2 \right) , \tag{39}$$

where the momentum conjugate to the vector potential is the generalized momentum. The standard analysis then leads to the Hamiltonian of the form

$$H = \sum_{\mathbf{k},\lambda}' \left(4\pi c^2 P^\dagger_{\mathbf{k},\lambda} P_{\mathbf{k},\lambda} + \frac{k^2}{4\pi} q^\dagger_{\mathbf{k},\lambda} q_{\mathbf{k},\lambda} \right) , \tag{40}$$

where the prime on the sum indicates that the sum is over only half the $\mathbf{k}$-space [9]. We may then introduce the operators in the familiar manner [9],

$$a_{\mathbf{k},\lambda} = \frac{1}{2} \{ q_{\mathbf{k},\lambda} + \frac{4\pi ic}{k} P_{\mathbf{k},\lambda} \} e^{ikct} \quad a^\dagger_{\mathbf{k},\lambda} = \frac{1}{2} \{ q_{\mathbf{k},\lambda} - \frac{4\pi ic}{k} P_{\mathbf{k},\lambda} \} e^{-ikct} \tag{41}$$

and the corresponding relations for the hermitian conjugate of the above. We identify, in the standard manner, the number operator

$$N_{\mathbf{k},\lambda} = \frac{k}{2\pi \hbar c} a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} . \tag{42}$$

In this manner, we may proceed to describe quantum electrodynamics in terms of the harmonic oscillators, using the variables $q_{\mathbf{k},\lambda}$ and the generalized momenta $P_{\mathbf{k},\lambda}$. We proceed through the standard analysis and arrive at the final result for the Hamiltonian,

$$H = \sum_{\mathbf{k},\lambda} \hbar ck (N_{\mathbf{k},\lambda} + \frac{1}{2}) , \tag{43}$$

describing the quanta of electrodynamics.

Proceeding further, we observe that the theory of coherent photon states [8] is of great interest and it is worthwhile investigating the consequences of quantum gravity for coherent states, in terms of the generalized uncertainty principle. For this purpose, we shall consider single level modes for the sake of simplicity in notation. The coherent states are defined by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle , \tag{44}$$

which satisfy the property, $a|\alpha\rangle = \alpha|\alpha\rangle$, where a is the annihilation operator defined in Eq.(41). The probability that a coherent state contains the n-quantum state is given by

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} . \tag{45}$$

Instead of the usual orthogonality relation, the coherent states obey the relation

$$\langle \alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha\beta)} \tag{46}$$

Employing the quadrature operator,

$$x_\lambda = \frac{1}{\sqrt{2}} \left(a\, e^{-i\lambda} + a^\dagger e^{i\lambda} \right), \qquad (47)$$

we obtain the result

$$\langle \alpha | x_\lambda^2 | \alpha \rangle = \frac{1}{2} \left(\alpha^2 e^{-2i\lambda} + \alpha^2 e^{2i\lambda} + 2|\alpha|^2 + 1, \right) \qquad (48)$$

for the deformed case, just as in the standard undeformed case [8]. In this manner, we obtain the familiar result for the variance,

$$\Delta q_\lambda \, \Delta q_{\lambda+\pi/2} = \frac{1}{2}. \qquad (49)$$

We accordingly find that the theory of coherent states when the generalized Heisenberg uncertainty principle prevails, is indeed described by the properties of the standard theory with no explicit consequences of the deformation.

IV. SUMMARY

We have studied the consequences of the generalized commutation relation leading to the generalized Heisenberg uncertainty relation, for a system described by quantum harmonic oscillators, referring to this case as a deformation described by the parameter β. Based on the fact that despite the deformation, the algebra of the creation and annihilation operators is no different from the standard undeformed algebra requires that the theory be described by a generalized momentum. We are thus able to describe the oscillators in a self-consistent formulation. We are able to show that the energy spectrum of the oscillators of a deformed system is no different from that of the standard quantum oscillators if the Hamiltonian is a quadratic sum of q and P. We have also studied the equations of motion in the deformed case, in order to point out how they differ from the standard equations of motion, and our results differ from that of ref.([1]). We employed the canonical Hamilton equations in order to derive the Liouville theorem, namely that the phase space density is a constant in time as the statistical system evolves. We have demonstrated that in contrast to the discussion in ref.([2]), the standard form of the Liouville theorem prevails in the deformed case when the canonical variables are described in terms of the generalized momentum. Finally, we have been able to study quantum electrodynamics and the theory of coherent state of photons to again show that there is no explicit consequence of quantum gravity related fundamental length in these cases.

We must point out the intimate connection between the undeformed harmonic oscillator and the generalized uncertainty relation. Not only is the oscillator algebra preserved by the generalization or deformation of the uncertainty relation, the energy spectrum is the same as of the standard form, and the ground state and excited state wave functions of the harmonic oscillator can be constructed in a straightforward manner. The generalized momentum P defined in Eq.(9) is related to the momentum p by a canonical transformation as revealed by Eqs.(29-39). We might thus regard the parameter β as transformed away by a canonical transformation.

The main purpose of this work is to address the question: can we reconcile the generalized uncertainty relation with the requirement that the oscillator algebra be preserved in its standard form? We are able to show that this can be accomplished if we introduce a generalized momentum. By extending the work of Camacho [3] in this manner, we have been able to demonstrate that the form of the generalized momentum is consistent with the generalized uncertainty relation incorporating a minimal length as well as with the standard form of the harmonic oscillator algebra.

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